DIFFRACTION OF A PLANE HYDROACOUSTIC WAVE ON THE BOUNDARY OF TWO ELASTIC PLATES

(DIFFRAKTSIIA PLOSKOI GIDROAKUSTICHESKOI VOLNY NA GRANITSE DVUKH UPRUGIKH PLASTIN)

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This paper is devoted to the diffraction of hydroacoustic waves by rectilinear inhomogeneities (cracks, seams of layers of different thickness) in an elastic layer. The case of low frequencies of the incident perturbation is considered here, i.e. of those frequencies for which the wavelength in the material of the layer is much greater than the thickness of the layer. The introduction of this restriction permits a transition from the contact problem for two media (liquid-elastic layer) to the boundary value problem for one medium, the liquid, on whose surface the boundary conditions are obtained on the basis of the vibration equations for an infinitely thin elastic plate. The mathematical foundation for the admissibility of such a transition (which is sufficiently evident physically) can be found in [1] for a case of similar nature, for an elastic layer submerged in a fluid.

Let us note that derivatives of the desired function of a higher order than the order of the equation itself enter into the boundary conditions of the problem under consideration. To insure uniqueness in the presence of possible discontinuities in the derivatives of the solution at the boundary, certain additional requirements at the points of these assumed discontinuities (contact conditions in the boundary conditions) have to be imposed on the solution of similar problems. Questions concerning the formulation of these problems are considered in [2]. The construction of the solutions of certain specific problems can be found in [2,3]; papers by G.D. Maliuzhinets* and V.Iu. Zavadskii** are devoted

* To the IV All-Union Conference on Acoustics (Moscow, 1958).

** To the First Symposium on Wave Diffraction (Odessa, 1960).

to the construction of these solutions. The problem Zavadskii solved (diffraction of a hydroacoustic wave from a semi-infinite plate located on the surface of a fluid) is most nearly similar to the problem to which this paper is devoted.

Later, only transverse vibrations of the layer will be taken into account. The consideration of certain processes [4] associated with longitudinal wave propagation in a layer will thus be excluded. As Krasil'nikov showed in his report to the Second Symposium on Wave Diffraction (Gor'kii, 1962), this simplification is justified since the energy level of processes associated with the presence of transverse waves in the layer dominates those for longitudinal waves in this case.

A general solution of the two-dimensional stationary problem of plane hydroacoustic wave diffraction at the boundary of two elastic plates with different elastic characteristics is constructed below for different contact conditions between the plates (seam, crack).

Notation

E – Young's modulus

σ – Poisson modulus

h – thickness of the layer

p – pressure

$\mathbf{v}$ – velocity of a fluid particle

ρ – density of the fluid

μ – surface density of the plate

ω – circular frequency

k – wave number in the fluid

t – time

ζ – transverse displacement of the plate

U – acoustic potential in the fluid

U_0 – incident wave

A – amplitude of the incident wave

φ_0 – angle at which the incident wave moves

Let two horizontal elastic plates be joined along a certain line and cover a half-space filled with fluid from above. A plane monochromatic acoustic wave whose direction of propagation is orthogonal to the line separating the plates (Fig. 1) arrives from the depths of the fluid. It is required to calculate the steady-state wave field which occurs as a result of reflection and diffraction of this wave.

Let us describe the processes in the fluid by the acoustic potential U and in the plate by the vertical displacement ζ. From mechanics we have the known relations

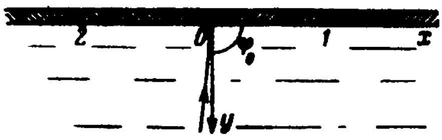

Fig. 1.

$$\Delta U + k^2U = 0, \quad \mathbf{v} = \operatorname{grad} U, \quad p = \rho i\omega U$$

$$-\frac{h^3E}{12(1-\sigma^2)}\Delta^2\zeta + \mu\omega^2\zeta = -p$$

The last relation describes the motion of the plate under the effect of the hydroacoustic pressure p. Since ζ is a point function on the plane, the operator Δ^2 should be taken over two variables in this expression as contrasted to the three-dimensional Laplace operator in the first formula. Let us agree to omit the time factor $e^{-i\omega t}$.

The location of the x and y axes is given in Fig. 1.

The z-axis is assumed to be directed along the line separating the plates (hence the dependence on the z coordinate vanishes). As a result, the problem of looking for two functions $U(x, y)$ and $\zeta(x)$ is obtained, where the former function is continuous in the whole domain of its arguments $(-\infty < x < \infty,\ 0 \leqslant y < \infty)$ and the second $(-\infty < x < \infty,\ x \neq 0)$ may have a jump type discontinuity at $x = 0$. These functions are sought under the following requirements.

1. The function U should satisfy the equation

$$\frac{\partial^2 U}{\partial x^2} + \frac{\partial^2 U}{\partial y^2} + k^2 U = 0 \qquad (-\infty < x < \infty,\ 0 < y < \infty)$$

2. The differences $V = U - U_0$ satisfy the principle of limiting absorption. ($U_0 = A \exp\left[i(\chi x - \sqrt{(k^2 - \chi^2}y)\right]$ is the incident wave, $\chi = -k\cos\varphi_0$, φ_0 is the angle at which the incident wave front moves).

3. The function ζ should satisfy the equation

$$-\frac{h_1^3E_1}{12(1-\sigma_1^2)}\frac{\partial^4\zeta}{\partial x^4} + \mu_1\omega^2\zeta = -\rho i\omega U(x, 0) \qquad (x > 0)$$

$$-\frac{h_2^3E_2}{12(1-\sigma_2^2)}\frac{\partial^4\zeta}{\partial x^4} + \mu_2\omega^2\zeta = -\rho i\omega U(x, 0) \qquad (x < 0)$$

4. The vertical displacements at the liquid-plate interface are continuous

$$i\omega\zeta = \left(\frac{\partial U}{\partial y}\right)_{y=0} \qquad (-\infty < x < +\infty,\ x \neq 0)$$

5. Certain contact relations which reflect the conditions on the

seam of the plates are satisfied for $\zeta(x)$ as $x \to \pm 0$, for example:

1) In the case of a seam

$$\zeta(-0) = \zeta(+0), \quad \frac{\partial \zeta(-0)}{\partial x} = \frac{\partial \zeta(+0)}{\partial x}$$ (continuity of the displacements and continuity of flexure)

$$\frac{h_2^3 E_2}{12(1-\sigma_2^2)} \frac{\partial^2 \zeta(-0)}{\partial x^2} = \frac{h_1^3 E_1}{12(1-\sigma_1^2)} \frac{\partial^2 \zeta(+0)}{\partial x^2}$$ (continuity of the bending moment)

$$\frac{h_2^3 E_2}{12(1-\sigma_2^2)} \frac{\partial^3 \zeta(-0)}{\partial x^3} = \frac{h_1^3 E_1}{12(1-\sigma_1^2)} \frac{\partial^3 \zeta(+0)}{\partial x^3}$$ (continuity of the bending stress)

2) In the case of an infinitely thin crack

$$\frac{\partial^2 \zeta(\pm 0)}{\partial x^2} = 0, \qquad \frac{\partial^3 \zeta(\pm 0)}{\partial x^3} = 0$$

The former of these relations requires the vanishing of a moment applied to the plate edges; the latter is the vanishing concentrated force on the plate edges.

To construct the solutions of the mentioned boundary-contact problems it is first necessary to satisfy the requirements 1-4. We shall designate the expression thus obtained for the potential as the general solution. As is seen, it contains four arbitrary constants which are determined from the boundary-contact relations on the joint of the plates.

Let us take a reflection-diffraction perturbation $V(x, y)$, obtained by subtraction of the incident wave from the whole field $V = U - U_0$, as the unknown function.

If the function ζ is eliminated from the boundary conditions 3 and 4, the problem reduces to the determination of the continuous function $V(x, y)$ $(-\infty < x < \infty,\ 0 \leqslant y < \infty)$ satisfying the requirements:

A) $$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + k^2 V = 0 \qquad (-\infty < x < \infty, 0 < y < \infty) \tag{1}$$

B) The principle of limiting absorption is satisfied for *V*.

C) $$\begin{aligned} L_1 V &\equiv \frac{\partial^5 V(x, 0)}{\partial x^4 zy} - \delta_1 \frac{\partial V(x, 0)}{\partial y} + \nu_1 V(x, 0) = iA_1 e^{i\varkappa x} \qquad (x > 0) \\ L_2 V &\equiv \frac{\partial^5 V(x, 0)}{\partial x^4 \partial y} - \delta_2 \frac{\partial V(x, 0)}{\partial y} + \nu_2 V(x, 0) = iA_2 e^{i\varkappa x} \qquad (x < 0) \end{aligned} \tag{2}$$

Here

$$\delta_l = \frac{\mu_l 12(1-\sigma_l^2)}{h_l^3 E_l}\omega^2, \qquad \nu_l = \rho\frac{12(1-\sigma_l^2)}{h_l^3 E_l}\omega^2 \qquad (l = 1, 2)$$

$$A_l = A\left(\varkappa^4\sqrt{k^2-\varkappa^2} - \delta_l\sqrt{k^2-\varkappa^2} + i\nu_l\right)$$

Let us solve the formulated problem by the method used by Maue in the problem of elastic wave diffraction from a half-plane [5]. Maliuzhinets [6,7] developed another method suitable for the solution. The Maue method (a modification of the Wiener-Hopf method) is more elementary and very convenient to our purposes. However, it should be noted that the Maliuzhinets method should be used when transferring to similar problems for angular domains since it is more general for such domains.

Let us separate the desired function V into two continuous components

$$V = V_1 + V_2$$

which separately satisfy the requirements (A) and (B) and as a sum, requirement (C). Let us formulate the boundary conditions for these components as follows:

$$L_1 v_1 = iA_1 e^{i\varkappa x} \quad (x > 0), \qquad L_2 v_1 = 0 \quad (x < 0) \tag{3}$$

$$L_1 v_2 = 0 \quad (x > 0), \qquad L_2 v_2 = iA_2 e^{i\varkappa x} \quad (x < 0) \tag{4}$$

Let us use the integral representation

$$V_1(x, y) = \frac{1}{2\pi i}\int_{-\infty}^{\infty} p_1(\lambda)\exp\left[i\left(\lambda x + \sqrt{k^2-\lambda^2}\,y\right)\right] / \, d\lambda \tag{5}$$

to look for the function V_1.

The requirements (A) and (B) for V_1 are thereby satisfied automatically. The radical $\sqrt{(k^2 - \lambda^2)}$ is considered positive on the segment $(-k, k)$ and the selection of its branch on the remaining sections of the integration contour is evident from Fig. 2 on which the contour is depicted by the solid line. The slits shown in Fig. 2 by dashes are made so that all the singularities of the function $p_1(\lambda)$ (except $\lambda = \chi$ about which more will be said below), located in the upper half-plane, would be to the right of the upper slit and those located in the lower half-plane would be to the left of the lower slit.

The boundary conditions (3) reduce to a system of integral equations for $p_1(\lambda)$

$$\frac{1}{2\pi i}\int_{-\infty}^{\infty} p_1(\lambda)\,(\lambda^4\sqrt{k^2-\lambda^2}-\delta_1\sqrt{k^2-\lambda^2}-i\nu_1)\,e^{i\lambda x}d\lambda = A_1e^{i\varkappa x} \qquad (x>0) \qquad (6)$$

$$\frac{1}{2\pi i}\int_{-\infty}^{\infty} (\lambda^4\sqrt{k^2-\lambda^2}-\delta_2\sqrt{k^2-\lambda^2}-i\nu_2)\,p_1(\lambda)\,e^{i\lambda x}\,d\lambda = 0 \qquad (x<0) \qquad (7)$$

The factors $\lambda^4\sqrt{(k^2-\lambda^2)}-\delta_l\sqrt{(k^2-\lambda^2)}-i\nu_l$ $(l = 1, 2)$ which grow as λ^5 at infinity generally cause divergence of the integrals in the left sides of equalities (6) and (7). Hence, these integrals should be understood in a certain generalized sense. Let us agree to identify the integral

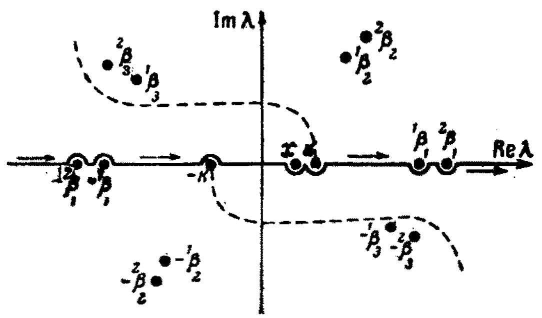

Fig. 2.

$$\int_{-\infty}^{\infty} f(\lambda)\,e^{i\lambda x}\,d\lambda \qquad (8)$$

(where $|f(\lambda)| < A|\lambda|^B$ for sufficiently large λ) with the integral of the same function over an arbitrary contour both of whose ends recede to infinity if the function $f(\lambda)$ remains analytic in the domain between the real axis and this contour. In order for the integral (8) to be made convergent for $x > 0$ because of the exponential, it is sufficient to bend the ends of the contour of integration in the upper half-plane in a suitable manner. For $x < 0$ the ends of the contour should be dropped correspondingly.

To satisfy equality (7) it is now sufficient to demand that the function

$$\Phi_-(\lambda) = p_1(\lambda^4\sqrt{k^2-\lambda^2}-\delta_2\sqrt{k^2-\lambda^2}-i\nu_2) \qquad (9)$$

be analytic in the lower half-plane. The other integral equation will be satisfied if

$$(\lambda^4\sqrt{k^2-\lambda^2}-\delta_1\sqrt{k^2-\lambda^2}-i\nu_1)\,p_1 = \frac{\Phi_+(\lambda)}{\Phi_+(\varkappa)}\frac{A_1}{\lambda-\varkappa} \qquad (10)$$

is satisfied, where $\Phi_+(\lambda)$ is analytic in the upper half-plane if it is considered that the contour of integration in (8) bypasses the pole $\lambda = \chi$ from below (Fig. 2). As a result, we arrive at the following boundary value problem for an analytic function (Riemann-Hilbert problem).

Let us look for two functions $\Phi_+(\lambda)$, analytic in the upper half-plane (more accurately, in the domain above the contour in Fig. 2) and $\Phi_-(\lambda)$, analytic in the lower-half-plane, whose ratio on the real axis is:

$$\frac{\Phi_+(\lambda)}{\Phi_-(\lambda)} = \frac{\Phi_+(\varkappa)(\lambda - \varkappa)}{A_1} \frac{\lambda^4 \sqrt{k^2 - \lambda^2} - \delta_1 \sqrt{k^2 - \lambda^2} - i\nu_1}{\lambda^4 \sqrt{k^2 - \lambda^2} - \delta_2 \sqrt{k^2 - \lambda^2} - i\nu_2} \tag{11}$$

To solve this problem, let us "factor" the functions

$$f_l(\lambda) = (\lambda^2 - k^2)(\lambda^4 - \delta_l) - i\nu \sqrt{k^2 + \lambda^2} \qquad (l = 1, 2)$$

i.e. let us represent each of them as the product of two factors one of which will be analytic in the upper half-plane and the other in the lower half-plane.

Simple algebraic analysis shows that the function $f_l(\lambda)$ is analytic on a two-sheeted Riemann surface, has ten roots on it of pairwise different signs. Three pairs of these roots are located on the main sheet and the two remaining pairs on the other sheet of the Riemann surface. Let $\pm\beta_{l1}$ $\pm\beta_{l2}$ $\pm\beta_{l3}$ ($l = 1, 2$) denote the roots on the main sheet, wherein two real roots $\pm\beta_{l1}$ are included. The principle of limiting absorption indicates that the contour of integration in (5) should bypass the positive real root $\pm\beta_{l1}$ from below and the negative, from above (Fig. 2). This can be seen if k is made slightly complex, i.e. put $k = k + i\varepsilon$. The root β_{l1} is then raised from the real axis into the upper half-plane.

Now let us separate factors corresponding to the roots

$$f_l(\lambda) = \varphi_l(\lambda) \prod_{s-1}^{3} (\lambda^2 - \beta_{ls}^2)$$

from the function $f_l(\lambda)$.

The functions $\varphi_l(\lambda)$ which remain after such a separation are analytic on the sheet under consideration do not have roots and tend to one at infinity.

Let us write down the Cauchy formula

$$\ln \varphi_l(\lambda) = \frac{1}{2\pi i} \int_C \frac{\ln \varphi_l(z)}{z - \lambda} dz$$

Here C is an arbitrary closed contour located on the sheet and enclosing the point $z = \lambda$. Expanding it, we arrive at the formula

$$\ln \varphi_l(\lambda) = \frac{1}{2\pi i}\int\limits_{C_-} \frac{\ln \varphi_l(z)}{z-\lambda}\,dz + \frac{1}{2\pi i}\int\limits_{C_+} \frac{\ln \varphi_l(z)}{z-\lambda}\,dz$$

where the contours C_- and C_+ enclose the slits (Fig. 3). Both improper integrals converge in the usual sense here because $\ln\varphi_i(z)$ tends to zero at infinity.

Hence, the first of the functions

$$\ln \varphi_l^+(\lambda) = \frac{1}{2\pi i}\int\limits_{C_+} \frac{\ln \varphi_l(z)}{z-\lambda}\,dz, \qquad \ln \varphi_l^-(\lambda) = \frac{1}{2\pi i}\int\limits_{C_-} \frac{\ln \varphi_l(z)}{z-\lambda}\,dz$$

is analytic outside the contour C_+ (and, in particular, in the upper half-plane) and the second is analytic in the lower half-plane. As a result we obtain the required partition $\varphi_l = \varphi_l^+\varphi_l^-$.

Now formula (11) can be represented as

$$A_1\frac{\Phi_+(\lambda)}{\Phi_+(\varkappa)}\,\frac{\varphi_2^+(\lambda)\,\Pi_2^+}{\varphi_1^+(\lambda)\,\Pi_1^+} = (\lambda-\varkappa)\,\Phi_-(\lambda)\,\frac{\varphi_1^-(\lambda)\,\Pi_1^-}{\varphi_2^-(\lambda)\,\Pi_2^-} \equiv F(\lambda) \tag{12}$$

$$\Pi_1^{\pm} = \prod_{s=1}^{3}(\lambda \pm \beta_{1s}), \qquad \Pi_2^{\pm} = \prod_{s=1}^{3}(\lambda \pm \beta_{2s})$$

The left side of this equality is a function analytic in the upper half-plane (more accurately, a function analytic above the contour of integration); the first is a function analytic in the lower half-plane. These functions transform continuously into each other on the boundary, hence, they can be considered as a single function $F(\lambda)$ analytic in the whole complex λ plane. Let us investigate the behavior of this function at infinity. Continuity of this function was stipulated in the definition of the function $V_1(x, y)$. To guarantee the continuity of this function on the x-axis in the representation (5), it is sufficient to demand the following estimate at infinity of the function $p_1(\lambda)$:

$$|p_1(\lambda)| \leqslant \frac{K}{|\lambda|^{1+\varepsilon}} \qquad (\varepsilon > 0)$$

In turn, this leads to the estimate

$$F(\lambda)| \leqslant K_1|\lambda|^{5-\varepsilon}$$

Then, according to the Liouville theorem, the function $F(\lambda)$ is a polynomial of fourth degree

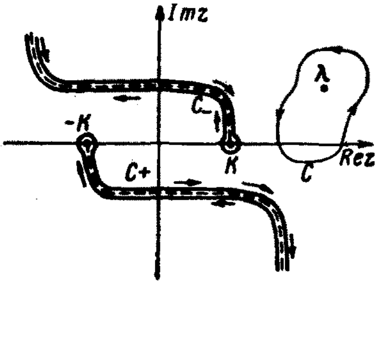

Fig. 3.

$$F(\lambda) = (a_1\lambda^3 + b_1\lambda^2 + c_1\lambda + d_1)(\lambda - \varkappa) + e_1 \tag{13}$$

The constant e_1 is evaluated from (12), (13) if we set $\lambda = \chi$

$$e_1 = \alpha\frac{A_1}{\bar{A}_1}, \quad \alpha = \frac{A}{\sqrt{k^2 - \varkappa^2}}\varphi_1^-(\varkappa)\varphi_2^+(\varkappa)\Pi_{12}(\varkappa), \Pi_{12}(\varkappa) = \prod_{s=1}^{3}(\varkappa - \beta_{1s})(\varkappa + \beta_{1s}) \tag{14}$$

Here $\bar{A}_1$ is the complex conjugate of A_1.

The other constants still remain arbitrary.

As a result we obtain for the function $V_1(x, y)$ (and analogously for $V_2(x, y)$)

$$V_1(x, y) = $$

$$= +\frac{1}{2\pi i}\int_{-\infty}^{\infty}\frac{\sqrt{k^2-\lambda^2}}{\lambda-\varkappa}\frac{[(a_1\lambda^3 + b_1\lambda^2 + c_1\lambda + d_1)(\lambda - \varkappa) + e_1]}{\varphi_1^-(\lambda)\varphi_2^+(\lambda)\Pi_{12}}e^{i(\lambda x + \sqrt{k^2-\lambda^2}y)}d\lambda \tag{15}$$

$$V_2(x, y) = $$

$$= -\frac{1}{2\pi i}\int_{-\infty}^{\infty}\frac{\sqrt{k^2-\lambda^2}}{\lambda-\varkappa}\cdot\frac{[(a_2\lambda^3 + b_2\lambda^2 + c_2\lambda + d_2)(\lambda - \varkappa) + e_2]}{\varphi_1^-(\lambda)\varphi_2^+(\lambda)\Pi_{12}}e^{i(\lambda x + \sqrt{k^2-\lambda^2}y)}d\lambda$$

Here

$$e_2 = \alpha\frac{A_2}{\bar{A}_2}, \quad \Pi_{12} = \prod_{s=1}^{3}(\lambda - \beta_{1s})(\lambda + \beta_{2s}) \tag{16}$$

and the contour of integration bypasses the pole $\lambda = \chi$ from above. Combining the integrals (15) and (16), we obtain a representation for the desired function U as a certain sum:

$$U = U_0 + U_1 + U_2 + W$$

$$U_1 = +\frac{1}{2\pi i}e_1\int_{-\infty}^{\infty}\frac{\sqrt{k^2-\lambda^2}}{(\lambda-\varkappa)\varphi_1^-\varphi_2^+\Pi_{12}}e^{i(\lambda x + \sqrt{k^2-\lambda^2}y)}d\lambda$$

$$U_2 = -\frac{1}{2\pi i}e_2\int_{-\infty}^{\infty}\frac{\sqrt{k^2-\lambda^2}}{(\lambda-\varkappa)\varphi_1^-\varphi_2^+\Pi_{12}}e^{i(\lambda x + \sqrt{k^2-\lambda^2}y)}d\lambda$$

$$W = \frac{1}{2\pi i}\int_{-\infty}^{\infty}\frac{a\lambda^3 + b\lambda^2 + c\lambda + d}{\varphi_1^-\varphi_2^+\Pi_{12}}e^{i(\lambda x + \sqrt{k^2-\lambda^2}y)}d\lambda \tag{17}$$

We call the components U_1 and U_2 the reflection-diffraction perturbations. The contour of integration in the expression for the former must bypass the pole $\lambda = \chi$ from below and in the latter, from above. These components individually satisfy the Helmholtz equation and the radiation condition, and as a sum, the boundary requirement (C). Moreover, each is continuous at the origin together with their derivatives up to the fourth order.

The diffraction perturbation W satisfies the zero boundary conditions and includes the discontinuities in the derivatives of U at the origin caused by the boundary-contact requirements. A system of four linear equations to find the values of the unknown constants a, b, c, d is obtained easily from the four boundary-contact equalities both for the case of a plate seam and for the case of a crack. In the seam case the coefficients a, b here turn out to be zero since U has continuous derivatives to the second order at the origin.

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